

# Uniqueness of $W^*$ -tensor Products

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## Abstract

In contrast to  $C^*$ -algebras, distinct  $C^*$ -norms on the algebraic tensor product of two  $W^*$ -algebras produce isomorphic  $W^*$ -tensor products.

AMS classification code: 46L06, 46L10

Keywords:  $W^*$ -tensor products

We use the notation and terminology of [C]. For  $W^*$ -tensor products of  $W^*$ -algebras we use [T].

In the sequel we give a list of some notation used in this paper.

1.  $\mathbb{K}$  denotes the field of real or the field of complex numbers. The whole theory is developed in parallel for the real and complex case (but the proofs coincide).
2. If  $f$  is a map defined on a set  $X$  and  $Y$  is a subset of  $X$  then  $f|Y$  denotes the restriction of  $f$  to  $Y$ .
3. If  $E, F$  are vector spaces in duality then  $E_F$  denotes the vector space  $E$  endowed with the locally convex topology of pointwise convergence on  $F$ , i.e. with the weak topology  $\sigma(E, F)$ .

4. If  $E$  is a Banach space then  $E'$  denotes its dual,  $E''$  its bi-dual, and  $E^\#$  its unit ball :

$$E^\# := \{ x \in E \mid \|x\| \leq 1 \}.$$

We put for every  $x \in E$ ,

$$j_E x : E' \longrightarrow \mathbb{K}, \quad x' \longmapsto \langle x, x' \rangle$$

and call the map  $j_E : E \longrightarrow E''$  the evaluation map of  $E$ . If  $F$  is a vector subspace of  $E$  then we set

$$F^0 := \{ x' \in E' \mid x'|_F = 0 \}.$$

If  $F$  is a vector subspace of  $E'$  then we define

$${}^0F := \{ x \in E \mid x' \in F \implies \langle x, x' \rangle = 0 \}.$$

5. Let  $E, F$  be Banach spaces and  $\varphi : E \longrightarrow F$  is a continuous linear map.  $\varphi$  is called an isometry if it preserves the norms and if it is surjective. We put  $Im \varphi := \varphi(E) = \{ \varphi x \mid x \in E \}$  and denote by

$$\varphi' : F' \longrightarrow E', \quad y' \longmapsto y' \circ \varphi$$

the transpose of  $\varphi$ .

6. If  $E$  is a  $C^*$ -algebra then we denote by  $Pr E$  the set of orthogonal projections of  $E$ . If in addition  $E$  is unital then we denote by  $1_E$  its unit. If  $F$  is a subset of  $E$  then we put

$$F^c := \{ x \in E \mid y \in F \implies xy = yx \}.$$

We put for all  $(x, x') \in E \times E'$

$$xx' : E \longrightarrow \mathbb{K}, \quad y \longmapsto \langle yx, x' \rangle,$$

$$x'x : E \longrightarrow \mathbb{K}, \quad y \longmapsto \langle xy, x' \rangle;$$

then  $xx', x'x \in E'$ .

7. If  $E$  is a  $W^*$ -algebra then  $\ddot{E}$  denotes its predual.

8.  $\odot$  denotes the algebraic tensor product. If  $E, F$  are  $C^*$ -algebras and if  $\alpha$  is a  $C^*$ -norm on  $E \odot F$  then  $E \otimes_\alpha F$  denotes the  $C^*$ -algebra obtained by the completion of  $E \odot F$  with respect to  $\alpha$ .

**Proposition 1** *Let  $E$  be a  $W^*$ -algebra and  $F$  a closed vector subspace of  $\ddot{E}$  such that  $xF \subset F$  and  $Fx \subset F$  for all  $x \in E$ .*

- a) *There is a  $p \in Pr E^c$  such that  $F = p\ddot{E}$  and  $F^0 = (1_E - p)E$ .*  
b) *For every  $x \in pE$  put*

$$\tilde{x} : F \longrightarrow \mathbb{K}, \quad a \longmapsto \langle x, a \rangle.$$

*Then  $\tilde{x} \in F'$  for all  $x \in pE$  and the map*

$$pE \longrightarrow F', \quad x \longmapsto \tilde{x}$$

*is an isometry of Banach spaces.*

- a) follows from [T] Theorem III.2.7 c).

- b) For  $a \in \ddot{E}$ ,

$$\langle x, a \rangle = \langle px, a \rangle = \langle x, ap \rangle = \langle \tilde{x}, ap \rangle,$$

so  $\tilde{x} \in F'$  and  $\|x\| = \|\tilde{x}\|$ . Let  $a' \in F'$  and put

$$y : \ddot{E} \longrightarrow \mathbb{K}, \quad a \longmapsto \langle a', pa \rangle.$$

Then  $y \in E$  and for  $a \in \ddot{E}$ ,

$$\langle py, a \rangle = \langle y, ap \rangle = \langle y, a \rangle,$$

so  $y = py \in pE$  and  $\tilde{y} = a'$ , i.e. the map is surjective. ■

**Proposition 2** *Let  $E$  be a  $C^*$ -algebra and  $F$  a closed vector subspace of  $E'$  such that  $xF \subset F$  and  $Fx \subset F$  for all  $x \in E$ .*

a) *There is a  $p \in \text{Pr}(E'')^c$  such that  $F = pE'$  and  $F^0 = (1_{E''} - p)E''$ .*

b) *The map*

$$pE'' \longrightarrow F', \quad x'' \longmapsto x''|_F$$

*is an isometry of Banach spaces.*

By [C] Corollary 1.3.6.5,  $\text{Im } j_E$  is dense in  $E''_{E'}$ , so  $x''F \subset F$  and  $Fx'' \subset F$  for all  $x'' \in E''$ . By [C] Theorem 6.3.2.1 b),  $E''$  is a  $W^*$ -algebra and the assertions follow from Proposition 1. ■

**Definition 3** *Let  $E, F$  be  $W^*$ -algebras. We define the (bilinear) maps*

$$(E \odot F) \times (\ddot{E} \odot \ddot{F}) \longrightarrow \ddot{E} \odot \ddot{F}, \quad (x \otimes y, a \otimes b) \longmapsto (x \otimes y)(a \otimes b) := (xa) \otimes (yb),$$

$$(\ddot{E} \odot \ddot{F}) \times (E \odot F) \longrightarrow \ddot{E} \odot \ddot{F}, \quad (a \otimes b, x \otimes y) \longmapsto (a \otimes b)(x \otimes y) := (ax) \otimes (by).$$

*and similarly the (bilinear) maps*

$$(E \odot F) \times (E' \odot F') \longrightarrow E' \odot F',$$

$$(E' \odot F') \times (E \odot F) \longrightarrow E' \odot F'.$$

**Proposition 4** *Let  $E, F$  be  $W^*$ -algebras,  $\alpha$  a  $C^*$ -norm on  $E \odot F$ , and*

$$G := \ddot{E} \bar{\otimes}_\alpha \ddot{F}$$

*the closure of  $\text{Im } j_{\ddot{E}} \odot \text{Im } j_{\ddot{F}}$  in  $(E \otimes_\alpha F)'$ .*

a) *There is a  $p \in \text{Pr}((E \otimes_\alpha F)'')^c$  such that  $G = p(E \otimes_\alpha F)'$  and such that the map*

$$p(E \otimes_\alpha F)'' \longrightarrow G', \quad x'' \longmapsto x''|_G$$

*is an isometry of Banach spaces.*

b) If  $j : E \otimes_\alpha F \longrightarrow (E \otimes_\alpha F)''$  denotes the evaluation map then  $\text{Im } j$  is a  $C^*$ -subalgebra of  $p(E \otimes_\alpha F)''$  generating it as a  $W^*$ -algebra.

a) For  $(u, v), (x, y) \in E \times F$  and  $(a, b) \in \ddot{E} \times \ddot{F}$ ,

$$\begin{aligned} \langle u \otimes v, (x \otimes y)((j_{\ddot{E}}a) \otimes (j_{\ddot{F}}b)) \rangle &= \langle (u \otimes v)(x \otimes y), (j_{\ddot{E}}a) \otimes (j_{\ddot{F}}b) \rangle = \\ &= \langle (ux) \otimes (vy), (j_{\ddot{E}}a) \otimes (j_{\ddot{F}}b) \rangle = \langle ux, j_{\ddot{E}}a \rangle \langle vy, j_{\ddot{F}}b \rangle = \\ &= \langle ux, a \rangle \langle vy, b \rangle = \langle u, xa \rangle \langle v, yb \rangle = \langle u, j_{\ddot{E}}(xa) \rangle \langle v, j_{\ddot{F}}(yb) \rangle = \\ &= \langle u \otimes v, j_{\ddot{E}}(xa) \otimes j_{\ddot{F}}(yb) \rangle, \end{aligned}$$

so

$$(x \otimes y)((j_{\ddot{E}}a) \otimes (j_{\ddot{F}}b)) = j_{\ddot{E}}(xa) \otimes j_{\ddot{F}}(yb) \in G.$$

It follows  $(x \otimes y)G \subset G$  and  $zG \subset G$  for all  $z \in E \otimes_\alpha F$ . Similarly  $Gz \subset G$  for all  $z \in E \otimes_\alpha F$ . By Proposition 2, there is a  $p \in \text{Pr}((E \otimes_\alpha F)'')^c$  such that  $G = p(E \otimes_\alpha F)'$  and such that the map

$$p(E \otimes_\alpha F)'' \longrightarrow G', \quad x'' \longmapsto x''|G$$

is an isometry of Banach spaces.

b Let  $(x, y) \in E \times F$  and  $\varepsilon > 0$ . There are  $a \in E^\#$  and  $b \in F^\#$  such that

$$\begin{aligned} \|x\| \|y\| - \varepsilon &< \langle x, a \rangle \langle y, b \rangle = \langle x, j_{\ddot{E}}a \rangle \langle y, j_{\ddot{F}}b \rangle = \\ &= \langle x \otimes y, (j_{\ddot{E}}a) \otimes (j_{\ddot{F}}b) \rangle = \langle j(x \otimes y), (j_{\ddot{E}}a) \otimes (j_{\ddot{F}}b) \rangle \leq \|j(x \otimes y)|G\|. \end{aligned}$$

Since  $\varepsilon$  is arbitrary,

$$\|j(x \otimes y)|G\| = \|x\| \|y\|,$$

so by a),  $j(x \otimes y) \in p(E \otimes_\alpha F)''$ . It follows  $j(E \odot F) \subset p(E \otimes_\alpha F)''$  and  $j(E \otimes_\alpha F) \subset p(E \otimes_\alpha F)''$ . Let

$$z \in (\text{Im } j_{\ddot{E}} \odot \text{Im } j_{\ddot{F}}) \cap {}^0(j(E \otimes_\alpha F)).$$

Then  $z|(E \otimes_\alpha F) = 0$  and so  $z = 0$ . Thus

$$G \cap {}^0(j(E \otimes_\alpha F)) = \{0\},$$

so by a),

$$\begin{aligned} {}^0(j(E \otimes_\alpha F)) &\subset (1_{(E \otimes_\alpha F)''} - p)(E \otimes_\alpha F)', \\ p(E \otimes_\alpha F)'' &\subset ({}^0(j(E \otimes_\alpha F)))^0, \\ p(E \otimes_\alpha F)'' &= ({}^0(j(E \otimes_\alpha F)))^0. \end{aligned}$$

Hence  $p(E \otimes_\alpha F)''$  is the closure of  $j(E \otimes_\alpha F)$  in  $(E \otimes_\alpha F)''_{(E \otimes_\alpha F)'}$  ([C] Proposition 1.3.5.4). By [C] Corollary 4.4.4.12 a),  $p(E \otimes_\alpha F)''$  is the  $W^*$ -subalgebra of  $(E \otimes_\alpha F)''$  generated by  $j(E \otimes_\alpha F)$ .  $\blacksquare$

**Definition 5** *We put (with the notation of Proposition 4)*

$$E \bar{\otimes}_\alpha F := p(E \otimes_\alpha F)''.$$

*It is a  $W^*$ -algebra with  $\ddot{E} \bar{\otimes}_\alpha \ddot{F}$  as predual.*

**Theorem 6** *If  $E, F$  are  $W^*$ -algebras and  $\alpha, \beta$  are  $C^*$ -norms on  $E \odot F$  then  $E \bar{\otimes}_\alpha F$  and  $E \bar{\otimes}_\beta F$  are isomorphic.*

We may assume  $\alpha \leq \beta$ . By [W] Proposition T.6.24, there is a surjective  $C^*$ -homomorphism

$$\varphi : E \otimes_\beta F \longrightarrow E \otimes_\alpha F.$$

Then

$$\varphi' : (E \otimes_\alpha F)' \longrightarrow (E \otimes_\beta F)'$$

preserves the norms ([C] Proposition 1.3.5.2). Thus  $\ddot{E} \bar{\otimes}_\alpha \ddot{F} = \ddot{E} \bar{\otimes}_\beta \ddot{F}$  and  $\varphi''$  is an isometry of Banach spaces. By [C] Corollary 6.3.2.3,  $\varphi''$  is a  $W^*$ -isomorphism.  $\blacksquare$

## REFERENCES

- [C] Constantinescu, Corneliu,  $C^*$ -algebras. Elsevir, 2001.
- [T] Takesaki, Masamichi, Theory of Operator Algebra I. Springer, 2002.
- [W] Wegge-Olsen, N. E.,  $K$ -theory and  $C^*$ -algebras. Oxford University Press, 1993.

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